1 Diagonal matrices

Matrix Λ is diagonal if all its off-diagonal elements are zero:

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_k) = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{pmatrix} \equiv \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} \tag{1}$$

where $\lambda_1, \ldots, \lambda_k$ are the diagonal elements of Λ. For simplicity, off-diagonal elements are often omitted.

Diagonal matrices have several important properties:

Linear independence: The fundamental property of diagonal matrices is linear independence of their columns, which holds if and only if all diagonal elements are non-zero. This directly follows from the fact that any linear combination of columns equaling zero requires all coefficients to be zero.

For any two distinct columns $i \neq j$ of a diagonal matrix Λ, their dot product is zero since they have non-overlapping non-zero elements:

$$\begin{pmatrix} \vdots \\ \lambda_i \\ 0 \\ \vdots \end{pmatrix}^{\mathsf{T}} \cdot \begin{pmatrix} \vdots \\ 0 \\ \lambda_j \\ \vdots \end{pmatrix} = 0$$

Basis set: For a diagonal matrix with non-zero diagonal elements, the columns form an orthogonal basis. Each column $\boldsymbol{b}_i$ contains exactly one non-zero element λ_i:

$$\mathcal{B}: \quad \boldsymbol{b}_1 = \begin{pmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \boldsymbol{b}_2 = \begin{pmatrix} 0 \\ \lambda_2 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \boldsymbol{b}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \lambda_k \end{pmatrix} \tag{2}$$

This orthogonal basis has a clear geometric interpretation — each basis vector aligns with a coordinate axis and has magnitude $|\lambda_i|$. When all $|\lambda_i| = 1$, the basis becomes orthonormal.

Scaling: A diagonal matrix Λ performs scaling transformations by independently scaling each coordinate by its corresponding diagonal element:

$$\Lambda \boldsymbol{v} = \begin{pmatrix} \lambda_1 v_1 \\ \vdots \\ \lambda_k v_k \end{pmatrix} \tag{3}$$

This represents stretching or compressing the space along each coordinate axis by factors $\lambda_1, \ldots, \lambda_k$.

$$\Lambda \boldsymbol{v} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 + 0 \\ \vdots \\ 0 + \lambda_k v_k \end{pmatrix}$$

Inverse matrix: For a diagonal matrix, its inverse is obtained by taking the reciprocal of each diagonal element, provided all diagonal elements are non-zero.

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_k)^{-1} = \mathrm{diag}\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k}\right) \tag{4}$$

Assuming, that the inverse of a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_k)$ is $\Lambda^{-1} = \mathrm{diag}\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k}\right)$, then:

$$\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\lambda_k} \end{pmatrix} = \begin{pmatrix} \lambda_1 \cdot \frac{1}{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \cdot \frac{1}{\lambda_k} \end{pmatrix} = I$$

Zero matrix O A diagonal matrix where all diagonal elements are zero:

$$O = \mathrm{diag}(0, \ldots, 0) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

Identity matrix I A diagonal matrix where all diagonal elements are one:

$$I = \mathrm{diag}(1, \ldots, 1) = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

Scalar matrix A diagonal matrix with constant diagonal elements:

$$\lambda I = \mathrm{diag}(\lambda, \ldots, \lambda) = \begin{pmatrix} \lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda \end{pmatrix}$$

Diagonal matrices are well suited for storing bases. If all diagonal elements are equal to one, the matrix is an identity matrix I and it stores orthonormal basis vectors of the standard basis.

In 3D space, the identity matrix I stores the standard basis vectors:

$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Commutativity: Diagonal matrices have the special property that they commute under multiplication:

$$\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1 \tag{5}$$

Moreover, diagonal matrices form a commutative group under multiplication.

For two diagonal matrices Λ and M:

$$\begin{aligned}\Lambda \mathrm{M} &= \begin{pmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_k\end{pmatrix}\begin{pmatrix}\mu_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \mu_k\end{pmatrix} = \begin{pmatrix}\lambda_1\mu_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_k\mu_k\end{pmatrix}\\ &= \begin{pmatrix}\mu_1\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \mu_k\lambda_k\end{pmatrix} = \begin{pmatrix}\mu_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \mu_k\end{pmatrix}\begin{pmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_k\end{pmatrix} = \mathrm{M}\Lambda\end{aligned}$$

Therefore, any order of multiplication of diagonal matrices results in the same diagonal matrix:

$$\Lambda \equiv \Lambda_1\Lambda_2 \ldots \Lambda_k = \Lambda_k \ldots \Lambda_2\Lambda_1 \tag{6}$$

Any diagonal matrix can be decomposed into a product of diagonal matrices, at least:

$$\Lambda = \Lambda I \ldots I$$

Eigenvalues and eigenvectors: For a diagonal matrix, the eigenvalues are precisely its diagonal elements, while the eigenvectors are the standard basis vectors of the space.

For a standard basis vector $\boldsymbol{e}_i$:

$$\Lambda \boldsymbol{e}_i = \begin{pmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_k\end{pmatrix} \cdot \boldsymbol{e}_i = \begin{pmatrix}0\\ \vdots\\ \lambda_i\\ \vdots\\ 0\end{pmatrix} = \lambda_i \begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix} = \lambda_i \boldsymbol{e}_i$$

so, $\boldsymbol{e}_i$ is an eigenvector of Λ with eigenvalue λ_i.

Eigenvalues and eigenvectors arise from the matrix equation $A\boldsymbol{v} = \lambda \cdot \boldsymbol{v}$, where A is a square matrix, $\lambda \in \mathbb{R}$ is an eigenvalue, and $\boldsymbol{v} \in \mathbb{R}^k$ is an eigenvector.

When viewing A as an operator, eigenvectors represent directions that maintain their orientation under the operation, being only scaled by the factor λ.